Birational Geometry Seminar 2023:

Fundamental groups of algebraic singularities.

Algebraic singularities: over complex numbers $\mathbb{C}$.

$(X;x)$ an algebraic singularity.

$L_{x,\varepsilon} := X \cap S(x,\varepsilon)$.

(Milnor 60's, Durfee 80's).

for ε small enough the diffeo class of $L_{x,\varepsilon}$ is independent of ε.

Local fundamental group: $\pi_1^{loc}(X;x) := \pi_1(L_{x,\varepsilon})$ for ε small.

Theorem (Mumford, 1961): Let $(X;x)$ be a normal surface sing.
Then $(X;x)$ is smooth $\iff$ $\pi_1^{loc}(X;x) \simeq \{1\}$.

Theorem (Grothendieck, 1968): The local fundamental groups of hypersurface sing of dim ≥ 3 are trivial.

Remark: Algebraic sing carry CW complex structures.
Then, local fundamental groups are f.p. groups.

Theorem (Kollár - Kapovich, 2011): For every f.p group G, there exists a complex projective surface S_G with snc singularities for which $\pi_1(S_G) \simeq G$. Voronoi - complexes.

Theorem (Kollár - Kapovich, 2011): For every f.p group G. there exists an isolated, normal, 3-fold sing $(X_G; 0)$ for which $\pi_1^{loc}(X_G; 0) \simeq G$.

Idea: Affine cone C_G over S_G.

$$\pi_1^{loc}(C_G; 0) \simeq G.$$

we smooth out the sing (outside 0) to construct an isolated sing with the same π_1

Rational & Cohen-Macaulay:

A group G is perfect if it has trivial abelianization $\Longleftrightarrow$ $H_1(G,\mathbb{Z}) = 0$.

superperfect + Schur multiplier is trivial. $\Longleftrightarrow$ $H_1(G;\mathbb{Z}) = H_2(G;\mathbb{Z}) = 0$.

$\mathbb{Q}$-perfect $\Longleftrightarrow$ $H_1(G;\mathbb{Q}) = 0$.

$\Updownarrow$

every abelian quotient is torsion

$\mathbb{Q}$-superperfect $\Longleftrightarrow$ $H_1(G;\mathbb{Q}) = H_2(G;\mathbb{Q}) = 0$.

Theorem (Kervaire, 1969): Let G be a f.p. $(\mathbb{Q})$-superperfect group & $n \geq 4$.

There exists a n-dimensional smooth $(\mathbb{Q})$-homology sphere M_G for which $\pi_1(M_G) \cong G$.

Remark: superperfect groups $\sim$ fundamental groups of homology spher

$\mathbb{Q}$-superperfect groups $\sim$ π_1 of $\mathbb{Q}$-homology sphere

Rational & CM sing:

Theorem (KK, 2011): Let $(X;x)$ be a rational sing. Then $\pi_1^{loc}(X;x)$ is a $\mathbb{Q}$-superperfect group. Every $\mathbb{Q}$-superperfect group is the π_1^{loc} of a rational isolated sing of dim ≥ 6.

Theorem (Kollár, 2012): TFAE for a f.p. group:

1) G is $\mathbb{Q}$-perfect,
2) G is the π_1^{loc} of an isolated CM sing of dim ≥ 3.

Question: What happens for the sing of the MMP?

log canonical

$(X;x)$ log terminal if there exists a log resolution

$$Y \xrightarrow{\varphi} X, \qquad \varphi^* K_X = K_Y + \sum_i \alpha_i E_i$$

where $\alpha_i < 1$.

$\alpha_i \leq 1$.

$\pi_1^{loc}(X;x)$ in the log terminal & log canonical?

Log terminal singularities:

Motto: log terminal singularities are local analogs of Fano varieties.

Fano: X has log terminal sing & $-K_X$ ample

Kobayashi 61: Smooth Fano varieties are simply connected.

Tsuji 88: (X, D) log smooth log Fano $(-(K_X+D)$ ample$)$ implies that X is simply connected.

90's several mathematicians (Zhang 94, McKernan - Keel 92, ...):

X is a Fano variety of dimension ≤ 3, then $\pi_1(X^{sm})$ is finite

Xu 2014: X Fano, then $\hat{\pi}_1(X^{sm})$ is finite; ← global finiteness

$(X;x)$ klt, then $\hat{\pi}_1^{loc}(X;x)$ is finite ← local finiteness

Tian - Xu 2015: global finiteness in dim $n-1 \Longrightarrow$ local finiteness in dim n.

Braun 2020: If $(X;x)$ klt, then $\pi_1^{loc}(X;x)$ is finite

for quotient sing $c(n) = n!$ if $n \geq 71$.

Theorem (Braun - Filipazzi - M - Svaldi, 2020): $\left\{ \begin{array}{l} n\text{-dim klt sing} \\ \text{with } S_{3n}\text{-actions} \end{array} \right\}$

There exists a constant $c(n)$, only depending on n, satisfying the following.

Let $(X;x)$ be a n-dimensional log terminal sing. There exists a s.e.s.

$$1 \longrightarrow A \longrightarrow \pi_1^{loc}(X;x) \longrightarrow N \longrightarrow 1$$

where A is finite abelian of rank $\leq n$, & N is finite of order at most $c(n)$.

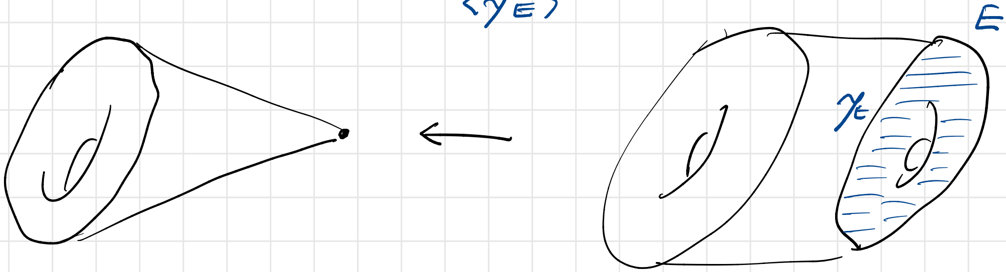
Log canonical singularities:

Example: Let $(X;x)$ be the cone over an elliptic curve.

there is a s.e.s.

$$1 \longrightarrow \underset{\substack{\| \\ \langle \gamma_E \rangle}}{\mathbb{Z}_m} \longrightarrow \pi_1^{loc}(X;x) \longrightarrow \mathbb{Z}^2 \longrightarrow 1$$

central.



Theorem (Figueroa - M, 2023): Let $(X,B;x)$ be a log canonical surface sing.

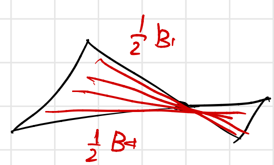
We have a short exact sequence

$$1 \longrightarrow N \longrightarrow \pi_1^{reg}(X,B;x) \longrightarrow G \longrightarrow 1$$

where N is solvable of length ≤ 2 & G is finite of order ≤ 6.

Furthermore, $\pi_1^{reg}(X,B;x)$ admits a presentation with at most 4 generators & 7 relations

If $\pi_1^{reg}(X,B;x)$ has a presentation with 4-gen & 7-rel. then $(X;x)$ is toric & $B = \frac{1}{2}B_1 + \dots + \frac{1}{2}B_4$



log canonical sing of higher dimension:

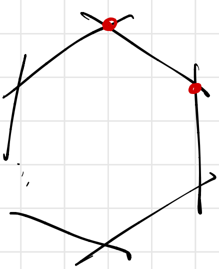
π_1 of closed 2-manifold with no boundary.

Theorem (Kollár, 2010): Let G be a surface group. There exists a 3-fold isolated lc sing $(X;x)$ and a s.e.s.

$$1 \longrightarrow \mathbb{Z}_m \longrightarrow \pi_1^{loc}(X;x) \longrightarrow G \longrightarrow 1$$

Theorem (Figueroa - M., 2023): Let E be a snc, projective, CY variety of dimension n. There exists a lc sing $(X;x)$ of dim $n+1$ for which $\pi_1^{loc}(X;x) \simeq \pi_1(E)$.

CY



$$K_E|_{\mathbb{P}^1} = K_{\mathbb{P}^1} + \{0\} + \{\infty\} \sim 0.$$

Idea (due to Kollár):

$$E \hookrightarrow \mathbb{P}^N$$

$$H^0(mH(-E)) \ni f_1, \dots, f_{N-n-1}$$

produce $Y \supseteq E$ of one dimension more $\dim Y = \dim E + 1$.

the normal bundle of E in Y is very negative.

$$\begin{tikzcd} E \arrow[d] & \subseteq & Y \arrow[d] \\ x & \in & X \end{tikzcd}$$

the fact that E is snc + CY $\Longrightarrow$ $(X; x)$ is lc of $\dim X = n+1$.

New input: We can perform birational modifications of E to make sure that $\pi_1(E) \simeq \pi_1^{loc}(X; x)$.

How to construct snc, CY, proj varieties?

$P \subseteq \mathbb{Q}^n$ smooth lattice polytope $\longleftrightarrow$ smooth proj toric variety $X(P) \hookrightarrow \mathbb{P}^N$.

Smooth polyhedral complex of dim n: finite category $\mathcal{P}$

objects: smooth polyhedra of dim $\leq n$

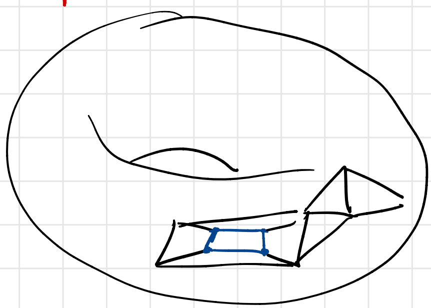
morphisms: lattice embeddings.

each vertex of $\mathcal{P}$ is contained in exactly $(n+1)$ maximal polyhedra.

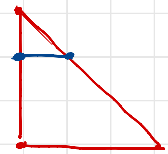
$\updownarrow$

n-dim proj snc CY varieties where each component is toric

T^2



2-dim smooth polyhedral complex homotopic to T^2.



Theorem (Figueroa - M, 2023): Let M be a 3-manifold that admits a smooth embedding in $\mathbb{R}^4$. There exists a 3-dim smooth polyhedral complex $\mathcal{P}_M$ homotopic to M.

$$M := \#_{i=1}^{r} (S^2 \times S^1)$$

Corollary: For every $r \geq 1$, there exists an isolated 4-dim lc sing $(X_r; o)$ for which $\pi_1^{loc}(X_r; o) \simeq F_r$.

smooth manifold of dim n $\xrightarrow{\text{topology + combinatorics}}$ smooth poly complex of dim n. $\xrightarrow{\text{toric geometry}}$ proj snc CY variety of dim n. $\xrightarrow{\text{alg geometry.}}$ lc sing of dim $n+1$